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A PROJECTIVE PROFINITE GROUP WHOSE SMALLEST EMBEDDING COVER IS NOT PROJECTIVE

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ABSTRACT

We constract a finitely generated projective group whose embedding cover is not projective. This solves Problem 23.16 of [FJ].

Introduction

A profinite group G has the **embedding property** if for each pair of epimorphisms (φ: $G \to A$, α: $B \to A$) where B is a finite quotient of G there exists an epimorphism γ: $G \to B$ such that $\alpha \circ \gamma = \varphi$. An epimorphism π: $E \to H$ of profinite groups such that E has the embedding property is an **embedding cover** of H. It is a **smallest embedding cover** of H if in addition, for each embedding cover φ: $G \to H$ there exists an epimorphism θ: $G \to E$ such that $\pi \circ \theta = \varphi$.

Haran and Lubotzky [HL] proved the existence and the uniqueness of the smallest embedding cover $E(H)$ of each finitely generated profinite group H. The case where H is finite was one of the essential ingredients in the decision procedure of the theory of perfect Frobenius fields [FJ, Cor. 23.19 and Thm. 25.11].

Haran and Lubotzky [HL] proved another important ingredient in the decidability procedure of the theory of perfect Frobenius fields: If a profinite group G

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has the embedding property, then so does its smallest projective cover $\widetilde{G}$ (also known as "universal Frattini cover") [FJ, Prop. 23.9]. Among others these led Haran and Lubotzky to state without proof the following statement as [HL, Cor. 2.12]: If H is a finitely generated profinite group, then $\widetilde{E(H)} = E(\widetilde{H})$. As it was not clear how to prove this corollary, [FJ] stated its truth and the truth of a related statement as an open problem:

PROBLEM ([FJ, Problem 23.16]): *Let H be a finitely generated profinite group.*
(a) *Is $E(H)$ projective whenever H is?*
(b) *Is $E(\widetilde{H})$ isomorphic to the smallest projective cover of $E(H)$?*

The goal of this note is to produce a finitely generated projective group H such that its smallest embedding cover $E(H)$ is not projective. In particular, $H = \widetilde{H}$ and therefore $E(\widetilde{H}) = E(H)$. On the other hand, the smallest projective cover $\widetilde{E(H)}$ is not isomorphic to $E(H)$, because the former is projective. So, both (a) and (b) are answered negatively, and Corollary 2.12 of [HL] is refuted.

We describe H in Proposition 1.2 and prove that although H is projective, $E(H)$ is not. In Section 2 we describe $E(H)$, by generators and relations. This description, which is independent of Section 1, proves again that $E(H)$ is not projective.

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1. The construction of H

Throughout this note we use l to denote a prime number and G_l for an l-Sylow group of a profinite group G. We also reserve p for an odd prime.

LEMMA 1.1: *Let N be a closed normal subgroup of a profinite group G. For each prime l choose an l-Sylow group G_l of G. If $N \neq 1$, then there exists an l such that $N \cap G_l \neq 1$.*

Proof: Choose a prime l that divides the order of N. Then its l-Sylow group $N \cap G_l$ is nontrivial. ∎

PROPOSITION 1.2: *Let $H = H_p \rtimes H_2$ be the profinite group defined by:*

$H_2 = \langle a, b \rangle$ *is the free pro-2-group on a, b,*

$H_p = \langle c \rangle \cong \mathbb{Z}_p$,

$c^a = c^{-1}$, $c^b = c$.

Then H is projective but its smallest embedding cover is not.

Proof: For each prime l, each l-Sylow group of H is l-free. Hence H is projective [FJ, Prop. 20.47]. We prove in four parts that the smallest embedding cover $\pi\colon E \to H$ is not projective.

PART A: H is generated by two elements, namely a and bc Indeed, choose a generator u for $\mathbb{Z}_p$ and a generator v for $\mathbb{Z}_2$. Since $bc = cb$ the map $(u, v) \mapsto (b, c)$ extends to an epimorphism $\mathbb{Z}_2 \times \mathbb{Z}_p \to \langle b, c\rangle$. As uv generates $\mathbb{Z}_2 \times \mathbb{Z}_p$, bc generates $\langle b, c\rangle$. Hence, $H = \langle a, b, c\rangle = \langle a, bc\rangle$, as claimed.

PART B: H does not have the embedding property Indeed, consider Klein's group $A_2 = \langle a_0, b_0\rangle$ of order r which is defined by the relations $a_0^2 = b_0^2 = 1$ and $a_0b_0 = b_0a_0$. The group A_2 acts on the cyclic group $A_p = \langle c_0\rangle$ of order p by $c_0^{a_0} = c_0^{-1}$ and $c_0^{b_0} = c_0$. The semidirect product $A = A_0 \rtimes A_2$ is a quotient of H via the map $(a, b, c) \to (a_0, b_0, c_0)$. Consider the epimorphism $\alpha\colon A \to A_2$ defined by $\alpha(a_0) = b_0$, $\alpha(b_0) = a_0$ and $\alpha(c_0) = 1$. Its kernel is $\langle c_0\rangle$. Consider the epimorphism $\eta\colon H \to A_2$ defined by $\eta(a) = a_0$, $\eta(b) = b_0$ and $\eta(c) = 1$.

Assume now that there exists an epimorphism $\theta\colon H \to A$ such that $\alpha \circ \theta = \eta$. Then $\theta(c) = c_0^i$, where i is relatively prime to p, and $\theta(a) = c_0^j b_0$. Apply θ to the relation $a^{-1}ca = a^{-1}$ to get $c_0^i = c_0^{-i}$. Hence, $p|2i$, a contradiction. So, θ does not exist and therefore H does not have the embedding property, as claimed.

PART C: E_p is abelian Indeed, let F be the free profinite group on two generators x, y. Use Part A to define an epimorphism $\varphi\colon F \to H$ by $\varphi(x) = a$ and $\varphi(y) = bc$. Let $U = \varphi^{-1}(\langle c\rangle)$ and $N = \mathrm{Ker}(\varphi)$. Then $F/U \cong H/\langle c\rangle \cong H_2$ is the free pro-2 group of rank 2. If U_0 is a closed normal subgroup of F such that F/U_0 is a pro-2 group, then its rank is 2 and the canonical epimorphism $F/U_0 \to F/U$ must be an isomorphism (a corollary of [FJ, Prop. 15.3]). Hence $U_0 = U$. Thus U is the smallest closed normal subgroup of F such that F/U is a pro-2 group. As such, U is a characteristic subgroup of F.

Let V be the smallest closed normal subgroup of U such that U/V is an abelian pro-p group. Then V is characteristic in U and therefore also in F. Since $U/N \cong \langle c\rangle \cong \mathbb{Z}_p$, the group N contains V. Let $\varphi'\colon F/V \to H$ be the epimorphism which φ induces.

Since F has the embedding property, so does F/V [FJ, Lemma 23.29]. Hence there exists an epimorphism $\gamma\colon F/V \to E$ such that $\pi \circ \gamma = \varphi'$. Note that U/V

is the p-Sylow group of F/V. So, γ maps U/V onto E_p. Since U/V is abelian, so is E_p.

PART D: Conclusion of the proof Since N/V is contained in U/V and the latter group is pro-p, the intersection of N/V with $(F/V)_2$ is trivial. Thus φ' is injective on $(F/V)_2$. Hence π is injective on $E_2 = \gamma((F/V)_2)$. Next note that the only primes which divide the order of F/V are 2 and p. By Part B, π is not injective. Hence, by Lemma 1.1, π is not injective on E_p. Since $\pi(E_p) = H_p \cong \mathbb{Z}_p$ and since E_p is abelian, this implies that E_p is not pro-p free. Conclude that E itself is not projective. ∎

2. The structure of $E(H)$

The existence of the two automorphisms of H_2 given by $(a,b) \to (b,a)$ and $(a,b) \to (a, a^{-1}b)$ forces the smallest embedding cover $E(H)$ to have two more generators for its p-Sylow group $E(H)_p$ such that a, b, and ab will have symmetric roles in their action on $E(H)_p$. Thus we prove that $E(H) = E(H)_p \rtimes E(H)_2$ where $E(H)_2 = H_2$, $E(H)_p = \langle c,d,e\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, and the action of $E(H)_2$ on $E(H)_p$ is given by (1) below.

If a profinite group G acts on a multiplicative abelian group A, we define for each $g \in G$ a homomorphism $\lambda_g\colon A \to A$ by $\lambda_g(a) = a^g a$.

We also let $G^2 = \langle g^2 | \, g \in G\rangle$. It is a closed normal subgroup of G. As $x^2 = 1$ for each $x \in G/G^2$, the latter group is abelian. In particular, if $G = \langle a,b\rangle$ is a pro-2 group of rank 2, then G^2 is the Frattini group of G and $G/G^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this case $G = G^2 a \,\dot\cup\, G^2 b \,\dot\cup\, G^2 ab \,\dot\cup\, G^2$.

LEMMA 2.1: *Let $G = G_p \rtimes G_2$ be a profinite group (possibly finite), where $G_p = \langle c,d,e\rangle$ is abelian, $G_2 = \langle a,b\rangle$, and the action of G_2 on G_p is given by*

$$c^a = c^{-1},\ d^a = d,\ e^a = e^{-1},\ c^b = c,\ d^b = d^{-1},\ e^b = e^{-1}. \tag{1}$$

Then

(a) *if $x \in G_p$ satisfies $x^2 = 1$, then $x = 1$,*
(b) *for each $x \in G_p$ we have $\langle x^2\rangle = \langle x\rangle$,*
(c) *for all $x,y \in G_2$, the action of x on G_p commutes with that of y,*
(d) *each $g \in G_2^2$ acts trivially on G_p,*
(e) *if $g \in G_2^2 a$, then $\mathrm{Im}(\lambda_g) = \langle d\rangle$,*
(f) *if $g \in G_2^2 b$, then $\mathrm{Im}(\lambda_g) = \langle c\rangle$,*

(g) *if* $g \in G_2^2 ab$, *then* $\mathrm{Im}(\lambda_g) = \langle e \rangle$, *and*
(h) $G_p = \langle c \rangle \times \langle d \rangle \times \langle e \rangle$.

Proof of (a): Let α be an element of $\mathbb{Z}_p$ such that $2\alpha = 1$. Then $x = x^{2\alpha} = 1$.

Proof of (b): For each $\beta \in \mathbb{Z}_p$ we have $x^\beta = x^{2\alpha\beta}$.

Proof of (c): As a and b generate G_2 it suffices to consider the case where $x = a$ and $y = b$. The action of a on $\{c, d, e\}$ commutes with that of b. Hence, so does its action on G_p.

Proof of (d): For each $h \in \{a, b\}$ and each $y \in \{c, d, e\}$ there exists $i \in \{\pm 1\}$ such that $y^h = y^i$. Hence, this is the case for each $h \in G_2$. It follows that h^2 acts trivially on $\{c, d, e\}$ and therefore also on G_p. Hence, each $g \in G_2^2$ acts trivially on G_p.

Proof of (e): Let $g = ya$ with $y \in G_2^2$. Then, by (d), $\lambda_g(c) = c^{ya}c = c^{-1}c = 1$, $\lambda_g(d) = d^2$, and $\lambda_g(e) = 1$. Hence, by (d), $\mathrm{Im}(\lambda_g) = \langle d^2 \rangle = \langle d \rangle$.

Proof of (f),(g): Similar to that of (e).

Proof of (h): Let $x \in \langle c \rangle \cap \langle d, e \rangle$. Then $x^a = x^{-1}$ and $x^b = x$, because x is an element of $\langle c \rangle$. On the other hand $x = d^\alpha e^\beta$, with $\alpha, \beta \in \mathbb{Z}_p$. Hence, $x^a = d^\alpha e^{-\beta}$ and $x^b = d^{-\alpha}e^{-\beta}$. Hence $d^{2\beta} = 1$, and therefore, by (a), $d^\beta = 1$. It follows that $e^{2\alpha} = 0$. As before, $e^\alpha = 1$. Hence $x = 1$. Conclude that $\langle c \rangle \cap \langle d, e \rangle = 1$.

Similarly $\langle d \rangle \cap \langle c, e \rangle = 1$ and $\langle e \rangle \cap \langle c, d \rangle = 1$. So, (h) is true. ∎

LEMMA 2.2: *Let G be as in Lemma 1 and let $\varphi \colon G \to H$ be a homomorphism. Then, with $H_p = \varphi(G_p)$ and $H_2 = \varphi(G_2)$ we have $H = H_p \rtimes H_2$.*

Proof: Since G_p is normal in G, the group H_p is normal in H. As $G = G_pG_2$, also $H = H_pH_2$. Finally, since $p \neq 2$, $H_p \cap H_2 = 1$. Hence, $H = H_p \rtimes H_2$. ∎

PROPOSITION 2.3: *Let $G = G_p \rtimes G_2$ be the profinite group defined by:*

$G_2 = \langle a, b \rangle$ is the free pro-2 group on a, b,

$G_p = \langle c, d, e \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$,

$$(1) \qquad c^a = c^{-1},\ d^a = d,\ e^a = e^{-1},\ c^b = c,\ d^b = d^{-1},\ e^b = e^{-1}.$$

Then G has the embedding property.

Proof: Let $\theta' \colon B \to A$ be an epimorphism of finite groups, and $\pi \colon G \to A$ and $\varphi' \colon G \to B$ two epimorphisms. We have to construct an epimorphism $\varphi \colon G \to B$

such that $\theta' \circ \varphi = \pi$. Obviously, with $\theta = \theta' \circ \varphi'$, it suffices to construct an automorphism φ of G such that $\theta \circ \varphi = \pi$.

For each $g \in G$ let $\bar{g} = \pi(g)$. By Lemma 2.2, $A = A_p \rtimes A_2$, where $A_2 = \pi(G_2) = \langle \bar{a}, \bar{b} \rangle$ and $A_p = \pi(G_p) = \langle \bar{c}, \bar{d}, \bar{e} \rangle$ satisfy the conditions of Lemma 2.1. In particular A_p is the unique p-Sylow subgroup of A and A_2 is a 2-Sylow subgroup of A. Thus $\theta(G_p) = A_p$ and $\theta(G_2)$ is conjugate to A_2. Hence, there exists $g \in G$ such that $\theta(G_2^g) = A_2$. Replace θ by conjugation by g followed by θ, if necessary, to assume that $\theta(G_2) = A_2$.

Use Gaschütz Lemma [FJ, Lemma 15.30] to choose generators α, β of G_2 such that $\theta(\alpha) = \bar{a}$ and $\theta(\beta) = \bar{b}$. As rank$(G_2) = 2$,

$$\{G_2^2\alpha, G_2^2\beta, G_2^2\alpha\beta\} = \{G_2^2 a, G_2^2 b, G_2^2 ab\}. \tag{2}$$

By Lemma 2.1, each of the groups $\mathrm{Im}(\lambda_\alpha)$, $\mathrm{Im}(\lambda_\beta)$, and $\mathrm{Im}(\lambda_\gamma)$ is cyclic. Now apply Lemma 2.1 on $A, \bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}$ instead of on G, a, b, c, d, e to conclude that

$$\begin{aligned} \theta(\mathrm{Im}(\lambda_\alpha)) &= \mathrm{Im}(\lambda_{\bar{a}}) = \langle \bar{d} \rangle, \\ \theta(\mathrm{Im}(\lambda_\beta)) &= \mathrm{Im}(\lambda_{\bar{b}}) = \langle \bar{c} \rangle, \\ \theta(\mathrm{Im}(\lambda_{\alpha\beta})) &= \mathrm{Im}(\lambda_{\bar{a}\bar{b}}) = \langle \bar{e} \rangle. \end{aligned}$$

Apply Gaschütz Lemma again to choose elements $\gamma, \delta, \varepsilon \in G_p$ such that

$$\begin{aligned} \mathrm{Im}(\lambda_\beta) &= \langle \gamma \rangle \text{ and } \theta(\gamma) = \bar{c}, \\ \mathrm{Im}(\lambda_\alpha) &= \langle \delta \rangle \text{ and } \theta(\delta) = \bar{d}, \\ \mathrm{Im}(\lambda_{\alpha\beta}) &= \langle \varepsilon \rangle \text{ and } \theta(\varepsilon) = \bar{e}. \end{aligned} \tag{3}$$

By Lemma 2.1 and by (2),

$$\{\langle \delta \rangle, \langle \gamma \rangle, \langle \varepsilon \rangle\} = \{\mathrm{Im}(\lambda_\alpha), \mathrm{Im}(\lambda_\beta), \mathrm{Im}(\lambda_{\alpha\beta})\} = \{\langle c \rangle, \langle d \rangle, \langle e \rangle\}.$$

In particular

$$\langle \delta \rangle \cap \langle \gamma \rangle = \langle \delta \rangle \cap \langle \varepsilon \rangle = \langle \gamma \rangle \cap \langle \varepsilon \rangle = 1. \tag{4}$$

Also, $\langle \gamma, \delta, \varepsilon \rangle = \langle c, d, e \rangle = G_p$. Hence, the map

$$(a, b, c, d, e) \mapsto (\alpha, \beta, \gamma, \delta, \varepsilon)$$

defines automorphisms φ_2: $G_2 \to G_2$ and φ_p: $G_p \to G_p$. If we prove that $\alpha, \beta, \gamma, \delta, \varepsilon$ satisfy

$$\gamma^\alpha = \gamma^{-1},\ \delta^\alpha = \delta,\ \varepsilon^\alpha = \varepsilon^{-1}, \gamma^\beta = \gamma,\ \delta^\beta = \delta^{-1},\ \varepsilon^\beta = \varepsilon^{-1}, \tag{5}$$

then φ_2 and φ_p can be combined to an automorphism φ: $G \to G$ such that $\theta \circ \varphi = \pi$.

Indeed, by (3), there exists $x \in G_p$ such that $\delta = x^\alpha x$. Hence, $\delta^\alpha = x^{\alpha^2} x^\alpha = xx^\alpha = x^\alpha x = \delta$. Similarly we argue for γ^β and $\varepsilon^{\alpha\beta}$ to prove:

$$\delta^\alpha = \delta,\ \gamma^\beta = \gamma,\ \varepsilon^{\alpha\beta} = \varepsilon. \tag{6}$$

By (3), $\gamma^\alpha\gamma = \lambda_\alpha(\gamma) \in \langle\delta\rangle$. By (6), by lemma 2.1(c), and by (3), $\gamma^\alpha\gamma = \gamma^{\beta\alpha}\gamma = \gamma^{\alpha\beta}\gamma = \lambda_{\alpha\beta}(\gamma) \in \langle\varepsilon\rangle$. Hence, by (4), $\gamma^\alpha\gamma = 1$ and therefore $\gamma^\alpha = \gamma^{-1}$. Similarly one proves the remaining relations of (3) to conclude the proof of the Proposition. ∎

THEOREM 2.4: *Let H be as in Proposition 1.2. Then the universal embedding cover of H is θ: $G \to H$, where G is the group defined in Proposition 2.3 and $\theta(a) = a$, $\theta(b) = b$, $\theta(c) = c$, and $\theta(d) = \theta(e) = 1$. Moreover, H is projective, while G is not.*

Proof: Observe first that $a, b, c, 1, 1$ satisfy the same relations in H as a, b, c, d, e in G. Hence, the map $(a, b, c, d, e) \to (a, b, c, 1, 1)$ extends to an epimorphism θ: $G \to H$. Since G has the embedding property (Proposition 2.3), θ is an embedding cover. We compare θ with the smallest embedding cover π: $E \to H$.

By definition, there exists an epimorphism φ: $G \to E$ such that $\pi \circ \varphi = \theta$. We have to prove that φ is an isomorphism.

By Lemma 2.2, $E = E_p \rtimes E_2$, where $E_p = \varphi(G_p)$ is the unique p-Sylow group of E and $E_2 = \varphi(G_2)$ is a 2-Sylow group of E. In particular $\pi(E_p) = \theta(G_p) = \langle c\rangle$ and $\pi(E_2) = \theta(G_2) = G_2$. Since, $E_2 = \langle\varphi(a), \varphi(b)\rangle$ and $G_2 = \langle a, b\rangle$ is the free pro-2 group on a, b, the restriction of φ to G_2 and the restriction of π to E are isomorphisms (a consequence of [FJ, Prop. 15.3]). So, without loss, identify $\varphi(a)$ with a, $\varphi(b)$ with b, and E_2 with G_2. The restriction of φ to G_2 and the restriction of π to E_2 become the identity maps.

Let $\bar{c} = \varphi(c)$, $\bar{d} = \varphi(d)$, and $\bar{e} = \varphi(e)$. By (1)

$$\bar{c}^a = \bar{c}^{-1},\ \bar{d}^a = \bar{d},\ \bar{e}^a = \bar{e}^{-1},\ \bar{c}^b = \bar{c},\ \bar{d}^b = \bar{d}^{-1},\ \bar{e}^b = \bar{e}^{-1}. \tag{7}$$

By Lemma 2.1(h), $E_p = \langle \bar{c} \rangle \times \langle \bar{d} \rangle \times \langle \bar{e} \rangle$. If we prove that each of the procyclic pro-p groups $\langle \bar{c} \rangle$, $\langle \bar{d} \rangle$, and $\langle \bar{e} \rangle$ are isomorphic to $\mathbb{Z}_p$, then the restriction of φ to G_p will also be an isomorphism and so φ will be proved to be an isomorphism, as desired. As $\pi(\bar{c}) = \theta(c) = c$ and $\langle c \rangle \cong \mathbb{Z}_p$ we have $\langle \bar{c} \rangle \cong \mathbb{Z}_p$. So, it remains to prove the same statement for $\bar{d}$ and $\bar{e}$.

It suffices to produce for each positive integer n epimorphisms of $\langle \bar{d} \rangle$ and $\langle \bar{e} \rangle$ on a cyclic group $A_p = \langle c_0 \rangle$ of order p^n. We start with $\bar{d}$.

Consider Klein's group $A_2 = \langle a_0, b_0 \rangle$ of order 4 which is defined by the relations $a_0^2 = b_0^2 = 1$ and $a_0 b_0 = b_0 a_0$. A_2 acts on A_p by $c_0^{a_0} = c_0^{-1}$ and $c_0^{b_0} = c_0$. Form the semidirect product $A = A_p \rtimes A_2$ and consider the epimorphism α: $A \to A_2$ defined by $\alpha(a_0) = b_0$, $\alpha(b_0) = a_0$ and $\alpha(c_0) = 1$. Its kernel is $\langle c_0 \rangle$.

Define a homomorphism η: $H \to A_2$ by $\eta(a) = a_0$, $\eta(b) = b_0$ and $\eta(c) = 1$. Since E has the embedding property and A is a finite quotient of H, and therefore of E, there exists an epimorphism κ: $E \to A$ such that $\alpha \circ \kappa = \eta \circ \pi$. In particular, as $\alpha(\kappa(a)) = \eta(\pi(a)) = a_0 = \alpha(b_0)$, there exists a positive integer i such that $\kappa(a) = c_0^i b_0$. Similarly, there exists a positive integer j such that $\kappa(b) = c_0^j a_0$. Also, $x = \kappa(\bar{c})$, $y = \kappa(\bar{d})$, and $z = \kappa(\bar{e})$ belong to $\mathrm{Ker}(\alpha) = \langle c_0 \rangle$. Hence, c_0 acts trivially on each of these elements. Observe that κ maps $E_p = \langle \bar{c}, \bar{d}, \bar{e} \rangle$ onto the p-Sylow group $\langle x, y, z \rangle = \langle c_0 \rangle$ of A. Apply κ on $\bar{c}^a = \bar{c}^{-1}$ to deduce that $x^{b_0} = x^{c_0^i b_0} = \kappa(\bar{c}^a) = x^{-1}$. On the other hand, as $c_0^{b_0} = c_0$ and $x \in \langle c_0 \rangle$, we have $x^{b_0} = x$. Hence $x = x^{-1}$ and therefore $x = 1$ (Lemma 2.1(a)). Similarly, starting from the relation $\bar{e}^a = \bar{e}^{-1}$ we deduce that $z = 1$. Hence, $\kappa(\langle \bar{d} \rangle) = \langle y \rangle = \langle c_0 \rangle$.

Now we handle $\bar{e}$. Define a homomorphism β: $A \to A_2$ by $\beta(a_0) = a_0$, $\beta(b_0) = a_0^{-1} b_0$, and $\beta(c_0) = 1$. Its kernel is $\langle c_0 \rangle$.

Since E has the embedding property, there exists an epimorphism λ: $E \to B$ such that $\beta \circ \lambda = \eta \circ \pi$. In particular there exists a positive integer i such that $\lambda(a) = c_0^i a_0$ and there exists a positive integer j such that $\lambda(b) = c_0^j a_0 b_0$. Also, $x = \lambda(\bar{c})$, $y = \lambda(\bar{d})$, and $z = \lambda(\bar{e})$ belong to $\mathrm{Ker}(\beta) = \langle c_0 \rangle$. Hence, c_0 acts trivially on each of these elements. Apply λ on the relation $\bar{c}^{ab} = \bar{c}^{-1}$ to conclude that $x^{b_0} = x^{a_0^2 b_0} = x^{c_0^i a_0 c_0^j a_0 b_0} = x^{-1}$. On the other hand, since $x \in \langle c_0 \rangle$ we have $x^{b_0} = x$. Hence $x = x^{-1}$ and therefore $x = 1$. Similarly, deduce from the relation $\bar{d}^{ab} = \bar{d}^{-1}$ that $y = 1$. Hence $\lambda(\bar{e}) = \langle z \rangle = \langle c_0 \rangle$, as desired.

Finally, observe that the Sylow subgroups of H are free. Hence H is projective [FJ, Prop. 20.47]. However, G_p is not free. Indeed, $\mathrm{cd}(G_p) = 3$. So, G is not projective. ∎

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